Report on Contract N7 ONR-28507, Task Order 7

Title: Fourier Transformers, Conformal Mapping, Entire Functions, and Asymptotic Solutions of Ordinary Differential Equations

February 1, 1954

1. R. W. McKelvey.

This work deals with the differential equation

(1) $u'' + \left\{\lambda^2 \phi(z) + \lambda\, \theta(z) + \sum_{n=0}^{\infty} \frac{\chi_n(z)}{\lambda^n}\right\} u = 0$ in which λ is a parameter that is large in absolute value. A point z at which $\phi(z)=0$, is called a <u>turning point</u>. When an equation (1) is considered in a z-region (z complex) that contains a turning point, the older methods for determining the functional forms of the solutions, and expressing them in terms of elementary functions, are inapplicable.

For equations (1) in which the zero of $\phi(z)$ is of the first order, a method for determining the forms of the solutions with explicitness that extends to an arbitrarily prescribed degree in $(1/\lambda)$ has been given by R. E. Langer. The basic medium of representation in that theory is the Bessel function. Langer's method does not admit of direct generalization to the case in which the zero of $\phi(z)$ is on the second order, because it involves integrals that then diverge. The purpose of this work has been to find the proper way of generalization. That has been found. It has been shown to lie in the use of the confluent hypergeometric function $M_{k,1/4}$ with a parameter k that depends upon λ. In terms of this medium of representation a quite complete theory has been constructed. The work was presented as a doctoral dissertation at the University of Wisconsin and will soon be submitted for publication. McKelvey has obtained his doctorate and has now terminated his connection with this project.

2. Anna Chandapillai.

The boundary problem defined by a differential equation and linear boundary conditions $u'' + \lambda p_1(x,\lambda) u' + \lambda^2 p_2(x,\lambda) u = 0$,

$$A_{i,1}(\lambda) u(\alpha) + A_{i,2}(\lambda) u'(\alpha) + A_{i,3}(\lambda) u(\beta) + A_{i,4}(\lambda) u'(\beta) = 0, \quad i = 1, 2,$$

in which each $p_j(x,\lambda)$ is a power series in $(1/\lambda)$ and each $A_{i,j}(\lambda)$ is a polynomial in $(1/\lambda)$, is well known when the interval $\alpha \le x \le \beta$ includes no

turning point of the differential equation. Such problems have been classified into categories known as "regular", "mildly irregular" and "highly irregular", and for each of these categories the distribution of Eigenwerte (characteristic values) has been determined. A few special problems of this type in which the interval contains a turning-point have been studied. They show that the Eigenwerte may be twice as numerous and quite differently distributed.

This investigator has started upon the assignment to study such problems generally, with a view toward their classification and a determination of the distributions of Eigenwerte that characterize them.

3. R. E. Langer.

Since the date of the last report (October 12, 1953) this investigator has been engaged in the rounding out of the theory outlined in the cited report. This concerns the asymptotic solution of differential equations of the type $u''' + \lambda p_1(z,\lambda) u'' + \lambda^2 p_2(z,\lambda) u' + \lambda^3 p_3(z,\lambda) u = 0$. in which λ is a large parameter and the $p_j(z,\lambda)$ are power series in $1/\lambda$. This theory is quite extensive. It will require some months yet to complete it. However, the completion is now assured.

An abstract of this theory has been submitted for presentation at the International Mathematical Congress to be held at Amsterdam, Holland next September.

4. Jacob Korevaar

Further results were obtained on numerical Tauberian theorems. The most general result obtained for power series was the following. Let $\sum a_n e^{-nu}$ be convergent for $u > 0$ to the sum $F(u)$, let $G(u)$ be analytic at $u = 0$, and let

$$|F(u) - G(u)| \leq \omega(u),$$

where $\omega(u) \downarrow 0$ as $u \downarrow 0$, on some interval $0 < u < d$. Let the a_n satisfy an inequality $a_n \geq -\varphi(n)$ $(n = 1, 2, \ldots)$, where $\varphi(n)$ is of the

form n^{α}, or more generally, $n^{\alpha}L(n)$, $L(n)$ a slowly oscillating function. Then there are numbers K_i $(K_4 > 1)$ such that

$$\left| \sum_1^n a_k - G(0) \right| \leq \rho(n) \qquad (n \geq K_2)$$

where

$$\rho(n) = \min_{p \geq K_2} \left\{ K_3 \, n\varphi(n)/p + K_4^p \, \omega(p/n) \right\}$$

if

$$\liminf_{u \downarrow 0} \; u \log \omega(u) > -\infty,$$

and

$$F(u) \equiv G(u), \qquad \rho(n) = K_5 \, \varphi(n)$$

if

$$\liminf_{u \downarrow 0} \; u \log \omega(u) = -\infty.$$

These estimates are essentially best possible. The proofs depend on best approximation in the L_1 sense by polynomials of given degree; the polynomials have to satisfy certain additional requirements dictated by the problem. The above result is contained in the paper: "Another numerical Tauberian theorem for power series", to appear in Proc. Nederl. Akad. Wetensch. 1954. This paper and its predecessor: "A very general form of Littlewood's theorem" (which was mentioned in the previous report) were presented to the American Mathematical Society at the Evanston meeting, November 27-28, 1953.

Corresponding numerical Tauberian theorems for Lambert series are much harder to establish. As a matter of fact, the analog for Lambert series of the above result for power series would prove the Riemann hypothesis. Whether conversely the Tauberian theorem follows from the Riemann hypothesis is at present under investigation.

The method used to prove the Tauberian theorem for power series can also be used to obtain Ostrowski's theorem on overconvergence of gap series, and certain related results obtained by Erdős and Piranian. These applications actually do not require the method in its most refined form. No time has been available to write up these results.

A short note entitled "Kloosterman's method in Tauberian theorems

for C_k summability" was submitted for publication in the Proc. Amer. Math. Soc.

Korevaar was invited to give the "Taft lectures" in mathematics at the University of Cincinnati on October 23-24, 1953. The titles of his talks were: I. Numerical Tauberian Theorems; II. Abstract Form of Tauberian Theory.

5. A. C. Schaeffer.

Let p be an odd prime. The fundamental domain of the group of transformations $\frac{a\tau+b}{c\tau+d}$ where a, b, c, d are rational integers such that ad - bc = 1, $b \equiv c \equiv 0$ modulo p, a+d even, b+c even is a Riemann surface R where genus can be written explicitely. Of fundamental importance for the study of the Dirichlet L series is the following question. If P_n is the divisor consisting of a pole of order n at the cycle $\tau = 1/p$ find all multiples of P_n. Some results have been obtained in this direction, and further work is being carried out.